

A FAMILY OF REPRESENTATIONS OF BRAID GROUPS ON SURFACES

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ABSTRACT. We propose a family of new representations of the braid groups on surfaces that extend linear representations of the braid groups on a disc such as the Burau representation and the Lawrence-Krammer-Bigelow representation.

1. INTRODUCTION AND PRELIMINARIES

1.1. Braid groups on surfaces. In the 1920's, E. Artin [2] began to study braids as mathematical objects and found a presentation of the group of braids of n strands under the binary operation concatenating corresponding strands of two braids. Since then, braids have served as convenient tools in various areas of science, especially in physics. Fox and Neuwirth [15] generalized the braid groups on arbitrary topological spaces M via configuration spaces and in this description, Artin's braid groups was on the plane \mathbb{R}^2 . Fadell and Neuwirth [13] found a fiber-bundle structure between configuration spaces, and Birman [10] observed that there is no interesting braid theory if the dimension of M is greater than 2. It is possible to consider braid groups on 1-dimensional complex [16]. But interesting braid groups are often obtained on 2-dimensional manifolds and we are also interested in these braid groups on surfaces in this article.

Let $\Sigma(g, p)$ denote a compact, connected, orientable 2-dimensional manifolds of genus g with p boundary components. We set $\Sigma = \Sigma(g, p)$. Let $\{z_1^0, \dots, z_n^0\}$ be a set of n preferred distinct points in Σ for $n \geq 0$ and let $\Sigma_n = \Sigma - \{z_1^0, \dots, z_n^0\}$. We call Σ_n the surface Σ with n punctures.

For integers $n, k \geq 0$, we consider three types of *configuration spaces* as follows: The space of k -tuples of distinct points in Σ_n denoted by

$$P_{n,k}(\Sigma) = \{(z_1, \dots, z_k) \in \Sigma_n \times \dots \times \Sigma_n \mid z_i \neq z_j \text{ for } i \neq j\},$$

the space of subsets of k elements in Σ_n denoted by

$$B_{n,k}(\Sigma) = \{\{z_1, \dots, z_k\} \subset \Sigma_n\},$$

and the space $B_{n;k}(\Sigma)$ of pairs of disjoint subsets of n elements and k elements in Σ denoted by

$$B_{n;k}(\Sigma) = \{(\{z_1, \dots, z_n\}, \{z_{n+1}, \dots, z_{n+k}\}) \mid z_i \in \Sigma, z_i \neq z_j \text{ for } i \neq j\}.$$

It is easy to see that $B_{n,k}(\Sigma) = P_{n,k}(\Sigma)/\mathbf{S}_k$ and $B_{n;k}(\Sigma) = P_{0,n+k}(\Sigma)/\mathbf{S}_n \times \mathbf{S}_k$ where the symmetric group \mathbf{S}_k acts on $P_{n,k}(\Sigma)$ by permuting components of a k -tuple and similarly $\mathbf{S}_n \times \mathbf{S}_k \subset \mathbf{S}_{n+k}$ acts on $P_{0,n+k}(\Sigma)$.

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The braid groups on a surface Σ are defined by the fundamental groups of configuration spaces. Choose a basepoint $\{z_{n+1}^0, \dots, z_{n+k}^0\}$ in $\partial\Sigma$ if $\partial\Sigma \neq \emptyset$. If Σ is closed, then place it anywhere in Σ_n . The *pure k -braid group on Σ_n* is defined and denoted by

$$\mathbf{P}_{n,k}(\Sigma) = \pi_1 \left(P_{n,k}(\Sigma), (z_{n+1}^0, \dots, z_{n+k}^0) \right).$$

Similarly, the *(full) k -braid group on Σ_n* is given by

$$\mathbf{B}_{n,k}(\Sigma) = \pi_1 \left(B_{n,k}(\Sigma), \{z_{n+1}^0, \dots, z_{n+k}^0\} \right),$$

and the *intertwining (n, k) -braid group on Σ* is given by

$$\mathbf{B}_{n;k}(\Sigma) = \pi_1 \left(B_{n;k}(\Sigma), (\{z_1^0, \dots, z_n^0\}, \{z_{n+1}^0, \dots, z_{n+k}^0\}) \right).$$

It is sometimes easier to understand if these groups are regarded as subgroups of $\mathbf{B}_{0,n+k}(\Sigma)$. The intertwining (n, k) -braid group $\mathbf{B}_{n;k}(\Sigma)$ is the preimage of $\mathbf{S}_n \times \mathbf{S}_k$ under the canonical projection: $\mathbf{B}_{0,n+k}(\Sigma) \rightarrow \mathbf{S}_{n+k}$. In addition, $\mathbf{B}_{n,k}(\Sigma)$ is the subgroup of $(n+k)$ -braids in $\mathbf{B}_{n;k}(\Sigma)$ that become trivial by forgetting the last k strands and $\mathbf{P}_{n,k}(\Sigma)$ is the subgroup of $(n+k)$ -braids in $\mathbf{B}_{n,k}(\Sigma)$ that are pure, that is, the induced permutation is trivial. If the surface Σ is the 2-disc D we will call the braid groups *classical*. For example, $\mathbf{B}_{0,n}(D)$ denotes the classical n -braid group studied by E. Artin.

In the 60's and 70's, presentations for braid groups on various surfaces are found: on the 2-sphere and the projective plane by Fadell and Van Buskirk [14, 30], on the torus by Birman [10], on all closed surfaces by Scott [29]. The study on braid groups on surfaces has been revived recently. González-Meneses [17] found new presentations of the braid groups on surfaces and Bellingeri [3, 4] found positive presentations of the braid groups $\mathbf{B}_{n,k}(\Sigma)$ for all surfaces Σ with or without boundaries. In this article, we are interested in braid groups that contain the classical braid groups as subgroups. So we will consider surfaces with nonempty boundary and will use Bellingeri's presentations.

We remark that boundary components of a surface can be traded with punctures when we consider braid groups. Let $\Sigma = \Sigma(g, p)$ and $\Sigma' = \Sigma(g, p+q)$. Then there are continuous maps $i : \Sigma_q \rightarrow \Sigma'$ and $j : \Sigma' \rightarrow \Sigma_q$ that are homotopy inverses each other. The induced maps $\tilde{i} : B_{n+q,k}(\Sigma) \rightarrow B_{n,k}(\Sigma')$ and $\tilde{j} : B_{n,k}(\Sigma') \rightarrow B_{n+q,k}(\Sigma)$ on configuration spaces are also homotopy inverses each other and induce isomorphisms \tilde{i}_*, \tilde{j}_* on fundamental groups [3, 28]. Therefore we may assume $\Sigma = \Sigma(g, 1)$ by treating all but one boundary component as punctures whenever we deal with a surface with nonempty boundary.

Throughout the article, we use Bellingeri's presentation [3] for the braid group $\mathbf{B}_{n,k}(\Sigma(g, 1))$ given as follows:

- Generators : $\sigma_1, \dots, \sigma_{k-1}, a_1, \dots, a_g, b_1, \dots, b_g, \zeta_1, \dots, \zeta_n$.
- Relations
 - BR1: $[\sigma_i, \sigma_j], \quad |i - j| \geq 2$
 - BR2: $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad |i - j| = 1$
 - CR1: $[a_r, \sigma_i], [b_r, \sigma_i], [\zeta_t, \sigma_i], \quad i > 1$
 - CR2: $[a_r, \sigma_1 a_r \sigma_1], [b_r, \sigma_1 b_r \sigma_1], [\zeta_t, \sigma_1 \zeta_t \sigma_1]$
 - CR3: $[a_r, \sigma_1^{-1} a_s \sigma_1], [a_r, \sigma_1^{-1} b_s \sigma_1], [b_r, \sigma_1^{-1} a_s \sigma_1], [b_r, \sigma_1^{-1} b_s \sigma_1], \quad r < s$
 $[a_r, \sigma_1^{-1} \zeta_u \sigma_1], [b_r, \sigma_1^{-1} \zeta_u \sigma_1], [\zeta_t, \sigma_1^{-1} \zeta_u \sigma_1], \quad t < u$
 - SCR: $\sigma_1 b_r \sigma_1 a_r \sigma_1 = a_r \sigma_1 b_r$.

The corresponding result for configuration spaces of pure braids by Fadell and Neuwirth can be generalized to show that the projection $B_{n,k}(\Sigma) \rightarrow B_{0,n}(\Sigma)$ is a fiber bundle with fiber $B_{n,k}(\Sigma)$. Except for S^2 and $\mathbb{R}P^2$, Guaschi and Gonçalves [18] completely determined when the short exact sequences of braid groups derived from Fadell-Neuwirth fibrations split. In particular, the short exact sequence derived from the fibration above

$$1 \rightarrow \mathbf{B}_{n,k}(\Sigma) \rightarrow \mathbf{B}_{n;k}(\Sigma) \rightarrow \mathbf{B}_{0,n}(\Sigma) \rightarrow 1$$

always splits for $k \geq 1$ if Σ has nonempty boundaries.

The braid groups are closely related to the mapping class groups. Birman [11] determined when surface braid group embeds into the corresponding mapping class group. In particular, if $\partial\Sigma \neq \emptyset$, $\mathbf{B}_{0,n}(\Sigma)$ embeds into the mapping class group on Σ_n and so an n -braid on Σ can be regarded as a homeomorphism of Σ that preserves the set of n punctures.

1.2. Representations of braid groups. The classical braid groups have various representations that can be as simple as taking exponent sums or taking induced permutations. The braid action on the punctured disk D_n gives rise a faithful representation into automorphism groups of free groups and a characterization of automorphisms coming from braid actions is possible. Each representation serves its own purpose. It is common to try to construct a linear representation to have a better understanding of a given group via matrices over a certain commutative ring and their multiplications.

For the classical braid groups, linear representations are abundant. Burau in 1936 and Gassner in 1961 discovered linear representations of $\mathbf{B}_{0,n}(D)$ and $\mathbf{P}_{0,n}(D)$, respectively. These representations are derived from braid actions on homologies of appropriate coverings of D_n . These representations take $(n-1) \times (n-1)$ matrices that can also be computed via Fox's free differential calculus on automorphisms of free groups mentioned above. The Burau representation is faithful for $n \leq 3$ but not faithful for $n \geq 5$ [5]. The faithfulness of Gassner representation is known only for $n \leq 3$.

Lawrence [24] discovered a family of linear representations of $\mathbf{B}_{0,n}(D)$ via a monodromy on a vector bundle over $P_{n,k}(D)$. Krammer [22] defined a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ -module V using *forks* and relations between them and he proved that the braid group acts on V faithfully for braid index 4 using algebraic and combinatorial argument. This representation is essentially the same as the one considered by Lawrence for $k = 2$ but uses configuration space $B_{n,2}(D)$ instead of $P_{n,2}(D)$ and is now called the *Lawrence-Krammer-Bigelow representation*. Bigelow reinterpreted this representation using covering spaces and covering transformation groups instead of vector bundles and local coefficients. Then the monodromy corresponds to the braid action on homology groups of covering spaces as the Burau representation and the Gassner representation were obtained. Bigelow [6] constructed a linear representation using homology group $H_2(\tilde{B}_{n,k}(D))$ of the covering space $\tilde{B}_{n,k}(D)$ whose covering transformation group is $\langle q \rangle \oplus \langle t \rangle$, and he proved that $\mathbb{R} \otimes V$ is isomorphic to $\mathbb{R} \otimes H_2(\tilde{B}_{n,k}(D))$. Moreover, Krammer [23] and Bigelow [6] independently proved that Lawrence-Krammer-Bigelow representation is faithful for all $n \geq 1$, and so the classical braid groups are linear. Furthermore, Bigelow and Budney [7] proved that the mapping class group of genus 2 surface has a faithful linear representation using the Lawrence-Krammer-Bigelow representation and

suitable branched covering. However, Paoluzzi and Paris showed that there is a difference between V and $H_2(\tilde{B}_{n,k})(D)$ as $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ -module for $n \geq 3$ and found basis for $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ -module $H_2(\tilde{B}_{n,k}(D))$ and so the exact definition of “Lawrence-Krammer-Bigelow representation” became little ambiguous.

For any $k \geq 1$, Bigelow in [9] considered the braid action on the Borel-Moore homology group $H_k^{BM}(\tilde{B}_{n,k}(D))$. He obtained a family of representations via the induced action on the image of $H_k^{BM}(\tilde{B}_{n,k}(D))$ in $H_k^{BM}(\tilde{B}_{n,k}(D), \partial\tilde{B}_{n,k}(D))$. For simplicity, we will directly consider the braid action on the free module $H_k^{BM}(\tilde{B}_{n,k}(D))$ whose basis can be easily described by forks to obtain a linear representation and we will call these representations *homology linear representations*. The Burau representation and the Lawrence-Krammer-Bigelow representation of $\mathbf{B}_{0,n}(D)$ are homology linear representations

$$\Phi_k : \mathbf{B}_{0,n}(D) \rightarrow \mathrm{GL} \left(\binom{n+k-2}{k}, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \right)$$

obtained from braid action on homologies of covers of $B_{n,k}(D)$ when $k = 1$ ($t = 1$ in this case) and $k = 2$, respectively [9].

In this article, we construct a family of representations of braid groups on surface with nonempty boundary that extend homological linear representations of the classical braid group. In §2, we first try to follow the way how the homology linear representations of $\mathbf{B}_{0,n}(D)$ was constructed via a covering of the configuration space $B_{n,k}(D)$. In the case of the disk, the braid action automatically commutes with covering transformation or braids act trivially on local coefficients in other words. However in case of surfaces of genus ≥ 1 , this condition forces the variable q equal 1 (see Lemma 2.6). Then the braid action becomes almost trivial. For example, if $k = 1$, the action of σ_i^2 is trivial. In order to get around this problem, we introduce the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ in §3 to replace $\mathbf{B}_{n,k}(\Sigma)$. As we mentioned earlier, this group is a semidirect product of $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$. Although the braid action does not preserve the local coefficient given by the $\mathbf{B}_{n,k}(\Sigma)$ factor, the $\mathbf{B}_{0,n}(\Sigma)$ factor of $\mathbf{B}_{n;k}(\Sigma)$ can adjust the coefficient so that the braid action becomes compatible. The coefficient ring for homology representations will also be extended in order to give a room to control the braid action in the expense of giving up its commutativity so that it becomes more interesting and still preserves coefficients. Eventually we obtain representations of braid groups on surfaces that extend homology linear representations of the classical braid group (see Theorem 3.2). And we explicitly compute the representations in the form of matrices using a geometric argument. We extend the intersection pairing between $H_k^{BM}(\tilde{B}_{n,k}(D))$ and its dual space $H_k(\tilde{B}_{n,k}(D), \partial\tilde{B}_{n,k}(D))$ and use bases for the two spaces that are described by “forks” and “noodles” (see Theorem 3.4).

In §4, we argue that the construction of our representations is natural and there are no other alternatives if one wants to obtain an extension of the homological representation using covers of the configuration space $B_{n,k}(D)$. We show that the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is the normalizer of $\mathbf{B}_{n,k}(\Sigma)$ in $\mathbf{B}_{0,n+k}(\Sigma)$ so that the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is such a group that is unique and maximal if we ignore a meaningless extension (see Theorem 4.2). The coefficient ring for our representations is the integral group ring of a quotient group of $\mathbf{B}_{n;k}(\Sigma)$. For $k \geq 3$, the quotient group is uniquely determined in order to extend homology linear representations of the classical braid group. For $k = 1, 2$, we use the simplest

one as long as it serves our purpose. Then we show that the braid action on the quotient group is virtually unique (see Theorem 4.4).

Our construction involving the group extension $\mathbf{B}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ is purely algebraic and does not carry a good geometric interpretation. As a result, some of useful geometric tools are not available. For example, the intersection pairing mentioned above is not invariant under the braid group action. This seems to make it difficult to discuss properties of our representations such as faithfulness and irreducibility. Although the corresponding representation of the classical braid group is faithful for $k = 2$ and irreducible for $k \leq 2$ [20, 31], both faithfulness and irreducibility for our representations are beyond the scope of the current article.

2. HOMOLOGY LINEAR REPRESENTATIONS

We first review the construction of homology linear representations of the classical braid group $\mathbf{B}_{0,n}(D)$ using the configuration space $B_{n,k}(D)$ and then discuss the difficulty in extending these homology linear representations to the braid group $\mathbf{B}_{0,n}(\Sigma)$ on a surface Σ with nonempty boundary. As we noted earlier, boundary components can be traded with punctures. From now on, we assume that Σ denotes a compact, connected, oriented surface with exactly one boundary component and that n and k are positive integers.

2.1. Homology linear representations of classical braid group. Let $\phi : \mathbf{B}_{n,k}(D) \rightarrow G$ be an epimorphism onto a group G . Consider the covering $p : \tilde{B}_{n,k}(D) \rightarrow B_{n,k}(D)$ corresponding to $\text{Ker } \phi$. Since the classical braid group embeds into the mapping class group of the punctured disk D_n , we may assume we have a homeomorphism $\bar{\beta} : B_{n,k}(D) \rightarrow B_{n,k}(D)$ for each $\beta \in \mathbf{B}_{0,n}$. By the lifting criteria, $\bar{\beta}$ lifts to $\tilde{\beta} : \tilde{B}_{n,k}(D) \rightarrow \tilde{B}_{n,k}(D)$ if and only if $\bar{\beta}_*(\text{Ker } \phi) \subset \text{Ker } \phi$. This is equivalent that there is an induced automorphism $\beta_\#$ on G such that $\beta_\# \phi = \phi \bar{\beta}_*$.

Now we consider *Borel-Moore homology* [12, 19] defined by

$$H_\ell^{BM}(\tilde{B}_{n,k}(D)) = \varprojlim H_\ell(\tilde{B}_{n,k}(D), p^{-1}(B_{n,k}(D) \setminus A))$$

where the inverse limit is taken over all compact subsets A of $B_{n,k}(D)$.

The middle-dimensional homology group $H_k^{BM}(\tilde{B}_{n,k}(D))$ is a free $\mathbb{Z}[G]$ -module of rank $\binom{n+k-2}{k}$ (see [9]) and $\tilde{\beta}$ induces a map $\tilde{\beta}_* : H_k^{BM}(\tilde{B}_{n,k}(D)) \rightarrow H_k^{BM}(\tilde{B}_{n,k}(D))$ such that

$$\tilde{\beta}_*(yc) = \beta_\#(y)\tilde{\beta}_*(c)$$

for $y \in G$ and $c \in H_k^{BM}(\tilde{B}_{n,k}(D))$. Thus the map $\tilde{\beta}_*$ is a $\mathbb{Z}[G]$ -module homomorphism if and only if $\beta_\#(y) = y$ for all $y \in G$ if and only if

$$(*) \quad \phi = \phi \bar{\beta}_* \quad \text{for all } \beta \in \mathbf{B}_{0,n}.$$

Notice that the condition $(*)$ also implies $\bar{\beta}_*(\text{Ker } \phi) \subset \text{Ker } \phi$. Here we need to know that the induced homomorphism $\tilde{\beta}_*$ depends only on the isotopy class of the homeomorphism β . In fact, since D has a boundary, we choose the basepoint $\{z_{n+1}^0, \dots, z_{n+k}^0\}$ of $B_{n,k}(D)$ in ∂D and then the isotopy preserves the basepoint and gives the same induced map $\tilde{\beta}_*$. Consequently, if we choose a group G and an epimorphism $\phi : \mathbf{B}_{n,k}(D) \rightarrow G$ satisfying $(*)$, we obtain a family of representation $\Phi_k : \mathbf{B}_{0,n}(D) \rightarrow \text{Aut}_{\mathbb{Z}[G]}(H_k^{BM}(\tilde{B}_{n,k}(D)))$ into the group of $\mathbb{Z}[G]$ -module

automorphisms on $H_k^{BM}(\tilde{B}_{n,k}(D))$ defined by

$$\Phi_k(\beta) = \tilde{\beta}_* : H_k^{BM}(\tilde{B}_{n,k}(D)) \rightarrow H_k^{BM}(\tilde{B}_{n,k}(D)).$$

Because we want to obtain a linear representation, the group G should be abelian. By the presentation given in §1.1, $\mathbf{B}_{n,k}(D)$ is generated by $\zeta_1, \dots, \zeta_n, \sigma_1, \dots, \sigma_{k-1}$. Suppose that $\phi : \mathbf{B}_{n,k}(D) \rightarrow G$ be an epimorphism satisfying the condition $(*)$ onto an abelian group G . Each generator σ_i of $\mathbf{B}_{0,n}(D)$ acts trivially on $\mathbf{B}_{n,k}(D)$ except

$$(\bar{\sigma}_i)_*(\zeta_i) = \zeta_i \zeta_{i+1} \zeta_i^{-1}, (\bar{\sigma}_i)_*(\zeta_{i+1}) = \zeta_i.$$

Then the condition $(*)$ implies that $\phi(\zeta_i) = \phi((\bar{\sigma}_i)_*(\zeta_{i+1})) = \phi(\zeta_{i+1})$. Hence for $k = 1$, G is a quotient of $\langle q \rangle$ and $\phi(\zeta_i) = q$ for $i = 1, \dots, n$. For $k \geq 2$, G is a quotient of $\langle q \rangle \oplus \langle t \rangle$ and $\phi(\zeta_i) = q, \phi(\sigma_j) = t$ for $i = 1, \dots, n$ and $j = 1, \dots, k-1$.

We define a group G_D and an epimorphism $\phi_D : \mathbf{B}_{n,k}(D) \rightarrow G_D$ depending only on k as follows:

$$\phi_D : \mathbf{B}_{n,k}(D) \rightarrow G_D = \begin{cases} \langle q \rangle & k = 1 \\ \langle q \rangle \oplus \langle t \rangle & k \geq 2 \end{cases}.$$

Theorem 2.1. [9, 24] *Let $\phi_D : \mathbf{B}_{n,k}(D) \rightarrow G_D$ be the epimorphism defined as above. Then there is a homomorphism*

$$\Phi_k : \mathbf{B}_{0,n}(D) \rightarrow \text{Aut}_{\mathbb{Z}[G_D]}(H_k^{BM}(\tilde{B}_{n,k}(D))).$$

In fact, Φ_1 is the Burau representation and Φ_2 is the Lawrence-Krammer-Bigelow representation.

2.2. Naive extension to braid groups on surfaces. Let Σ be a surface of genus $g \geq 1$ with nonempty boundary. Besides the two reasons mentioned at the end of §1.1, there are one more reason that makes the assumption $\partial\Sigma \neq \emptyset$ necessary. Suppose that $\partial\Sigma = \emptyset$ and $\beta \in \mathbf{B}_{0,n}(\Sigma)$ uniquely determines the isotopy class of a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$. Then we must choose the basepoint $\{z_{n+1}^0, \dots, z_{n+k}^0\}$ in the interior of Σ . We can easily find a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$ that is isotopic to the identity via an isotopy that does not preserve the basepoint. Then β represents the identity element in $\mathbf{B}_{0,n}(\Sigma)$ but $\tilde{\beta}_* : H_k^{BM}(\tilde{B}_{n,k}(\Sigma)) \rightarrow H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ may be nontrivial. Thus no representation can be obtained in this way if $\partial\Sigma = \emptyset$.

We need to define when we say that a representation of the braid group $\mathbf{B}_{0,n}(\Sigma)$ extends homology linear representations of the classical braid groups.

Definition 2.2. *Given a ring R , let M be an R -module on which the braid group $\mathbf{B}_{0,n}(\Sigma)$ acts as R -module isomorphisms. The R -module M is an extension of homology linear representations of the classical braid groups $\mathbf{B}_{0,n}(D)$ if there exists a $\mathbb{Z}[G_D]$ -submodule M' of M such that*

- (i) *It is invariant under the action by the subgroup $\mathbf{B}_{0,n}(D)$ of $\mathbf{B}_{0,n}(\Sigma)$;*
- (ii) *For some $k \geq 1$, R contains $\mathbb{Z}[G_D]$ as a subring and there is a $\mathbb{Z}[G_D]$ -isomorphism from $H_k^{BM}(\tilde{B}_{n,k}(D))$ to M' that commutes with the $\mathbf{B}_{0,n}(D)$ action.*

As in the classical braid cases, we have to look at the action of $\mathbf{B}_{0,n}(\Sigma)$ on $\mathbf{B}_{n,k}(\Sigma)$. The following lemma helps us to observe the action we desire.

Lemma 2.3. [11, 13, 18] *Let $\pi_n : B_{n;k}(\Sigma) \rightarrow B_{0,n}(\Sigma)$ be the projection onto the first n coordinates. Then the space $B_{n;k}(\Sigma)$ is a fiber bundle with fiber $B_{n,k}(\Sigma)$ and the induced short exact sequence*

$$1 \longrightarrow \mathbf{B}_{n,k}(\Sigma) \longrightarrow \mathbf{B}_{n;k}(\Sigma) \xrightarrow{(\pi_n)_*} \mathbf{B}_{0,n}(\Sigma) \longrightarrow 1$$

splits for all $k \geq 1$.

This lemma shows that how to decompose a braid $\beta \in \mathbf{B}_{n;k}(\Sigma)$ into a product $\beta = \beta_1 \beta_2$ for $\beta_1 \in \mathbf{B}_{0,n}(\Sigma)$ and $\beta_2 \in \mathbf{B}_{n,k}(\Sigma)$. Let $\iota : \mathbf{B}_{0,n}(\Sigma) \rightarrow \mathbf{B}_{n;k}(\Sigma)$ be the splitting map. Then the above lemma shows that $\mathbf{B}_{n;k}(\Sigma)$ can be generated by the following two sets:

$$X_1 = \{\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\mu}_1, \dots, \bar{\mu}_g, \bar{\lambda}_1, \dots, \bar{\lambda}_g\},$$

$$X_2 = \{\sigma_1, \dots, \sigma_{k-1}, \zeta_1, \dots, \zeta_n, \mu_1, \dots, \mu_g, \lambda_1, \dots, \lambda_g\}$$

where the generators in X_1 are the images of generators in $\mathbf{B}_{0,n}(\Sigma)$ under the inclusion map ι .

Then the action of $\mathbf{B}_{0,n}(\Sigma)$ on $\mathbf{B}_{n,k}(\Sigma)$ is equivalent to the conjugate action in $\mathbf{B}_{n;k}(\Sigma)$ if we regard these two groups as subgroups of $\mathbf{B}_{n;k}(\Sigma)$. The following lemma shows how $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ and its proof is straightforward and omitted.

Lemma 2.4. *Each generator of $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ as follows.*

(1) For $1 \leq i \leq n-1$,

$$(\bar{\sigma}_i)_*(\zeta_t) = \begin{cases} \zeta_i \zeta_{i+1} \zeta_i^{-1} & t = i \\ \zeta_i & t = i+1 \end{cases}$$

(2) For $1 \leq r \leq g$,

$$\begin{aligned} (\bar{\mu}_r)_*(\zeta_1) &= \mu_r \zeta_1 \mu_r^{-1} \\ (\bar{\mu}_r)_*(\mu_s) &= \begin{cases} \mu_r \zeta_1 \mu_r \zeta_1^{-1} \mu_r^{-1} & s = r \\ [\mu_r, \zeta_1] \mu_s [\mu_r, \zeta_1]^{-1} & r < s \end{cases} \\ (\bar{\mu}_r)_*(\lambda_s) &= \begin{cases} \lambda_r \mu_r \zeta_1^{-1} \mu_r^{-1} & s = r \\ [\mu_r, \zeta_1] \lambda_s [\mu_r, \zeta_1]^{-1} & r < s \end{cases} \end{aligned}$$

(3) For $1 \leq r \leq g$,

$$\begin{aligned} (\bar{\lambda}_r)_*(\zeta_1) &= \lambda_r \zeta_1 \lambda_r^{-1} \\ (\bar{\lambda}_r)_*(\mu_s) &= \begin{cases} \lambda_r \zeta_1 \lambda_r^{-1} \mu_r \zeta_1 \lambda_r \zeta_1^{-1} \lambda_r^{-1} & s = r \\ [\lambda_r, \zeta_1] \mu_s [\lambda_r, \zeta_1]^{-1} & r < s \end{cases} \\ (\bar{\lambda}_r)_*(\lambda_s) &= \begin{cases} \lambda_r \zeta_1 \lambda_r \zeta_1^{-1} \lambda_r^{-1} & s = r \\ [\lambda_r, \zeta_1] \lambda_s [\lambda_r, \zeta_1]^{-1} & r < s \end{cases} \end{aligned}$$

(4) All other generators act trivially.

We can find the presentation for $\mathbf{B}_{n;k}(\Sigma)$ using the above lemma as follows.

Lemma 2.5. *The braid group $\mathbf{B}_{n;k}(\Sigma)$ admits the following presentation*

• *Generators:*

$$\begin{aligned} X_1 &= \{\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\mu}_1, \dots, \bar{\mu}_g, \bar{\lambda}_1, \dots, \bar{\lambda}_g\} \\ X_2 &= \{\sigma_1, \dots, \sigma_{k-1}, \zeta_1, \dots, \zeta_n, \mu_1, \dots, \mu_g, \lambda_1, \dots, \lambda_g\} \end{aligned}$$

• *Relations:*

- (i) BR1 through SCR among generators in X_1
- (ii) BR1 through SCR among generators in X_2
- (iii) $\bar{x}^{-1} y \bar{x} = (\bar{x}_*)(y)$ for all $\bar{x} \in X_1, y \in X_2$

where the relations *BR1* through *SCR* are from Bellingeri's presentation in §1.1 and the action by \bar{x}_* is given in Lemma 2.4.

Proof. By Lemma 2.3, the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$ is a semidirect product of the normal subgroup $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$ where $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ by conjugation as shown in Lemma 2.4. Then it is easy to show that the semidirect product $\mathbf{B}_{n;k}(\Sigma)$ admits the desired presentation. \square

For surfaces, the condition $(*)$ implies an undesirable consequence as shown in the following lemma.

Lemma 2.6. *Let $\phi : \mathbf{B}_{n,k}(\Sigma) \rightarrow G$ be an epimorphism satisfying $\phi = \phi\bar{\beta}_*$ for any $\beta \in \mathbf{B}_{n,k}(\Sigma)$. Then for $i = 1, \dots, n$,*

$$\phi(\zeta_i) = 1.$$

Proof. As seen earlier, the hypothesis on ϕ implies that for all $y \in G$ and $r = 1, \dots, g$,

$$(\mu_r)_\#(y) = y.$$

But by Lemma 2.4 (2), we have

$$\begin{aligned} (\mu_r)_\#(\lambda_r) &= \phi((\bar{\mu}_r)_*(\lambda_r)) = \phi(\lambda_r \mu_r \zeta_1^{-1} \mu_r^{-1}) \\ &= \phi(\lambda_r) \phi((\bar{\mu}_r)_*(\zeta_1^{-1})) = \phi(\lambda_r) (\mu_r)_\#(\phi(\zeta_1^{-1})) = \phi(\lambda_r) \phi(\zeta_1^{-1}). \end{aligned}$$

Since $(\mu_r)_\#(\phi(\lambda_r)) = \phi(\lambda_r)$ by hypothesis, $\phi(\zeta_1) = 1$ and so $\phi(\zeta_i) = 1$ for all $1 \leq i \leq n$ by Lemma 2.4 (1). \square

The previous lemma says that the condition $(*)$ forces to set $q = 1$ in the group G_D . Thus $\mathbb{Z}[G_D]$ cannot be a subring of $\mathbb{Z}[G]$ and so a naive attempt to obtain a representation of the braid group $\mathbf{B}_{0,n}(\Sigma)$ using a covering of $B_{n,k}(\Sigma)$ corresponding to any epimorphism $\phi : \mathbf{B}_{n,k}(\Sigma) \rightarrow G$ cannot give an extension of any homology linear representation of the classical braid groups.

3. A FAMILY OF PROPOSED REPRESENTATIONS

As we have seen in the previous section, we are forced to take a rather small covering of $B_{n,k}(\Sigma)$ in order that the condition $(*)$ is satisfied, that is, the braid action commutes with covering transformations so that it preserves the coefficient. A remedy that we propose in this article is to use the same configuration space $B_{n,k}(\Sigma)$ with an extended coefficient ring so that we have some room to adjust coefficients to make the braid action compatible with the coefficients. This remedy is a reasonable thing to do if we hope to construct an extension of homology linear representations of the classical braid groups. Indeed, we successfully obtain an extension that seems the most general among ones obtained from coverings of $B_{n,k}(\Sigma)$.

3.1. Existence of extension of homology linear representation. We first consider the intertwining braid group $\mathbf{B}_{n;k}(\Sigma)$. Note that $\mathbf{B}_{n;k}(\Sigma)$ is a candidate of group extension of $\mathbf{B}_{n,k}(\Sigma)$ by Lemma 2.3, and $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n;k}(\Sigma)$ by right multiplication and acts on $\mathbf{B}_{n,k}(\Sigma)$ by conjugate because $\mathbf{B}_{n;k}(\Sigma)$ is the semidirect product of $\mathbf{B}_{0,n}(\Sigma)$ and $\mathbf{B}_{n,k}(\Sigma)$.

Let H_Σ be the abstract group depending only on k which admits the following presentation: for $k \geq 2$,

- Generators : $q, t, \bar{m}_1, \dots, \bar{m}_g, \bar{\ell}_1, \dots, \bar{\ell}_g, m_1, \dots, m_g, \ell_1, \dots, \ell_g$
- Relations : All generators commute except for the following

$$[m_r, \ell_r] = t^2, \quad [\bar{m}_r, \ell_r] = [m_r, \bar{\ell}_r] = q$$

Let $\psi_\Sigma : \mathbf{B}_{n,k}(\Sigma) \rightarrow H_\Sigma$ be the epimorphism onto a group H_Σ defined by

$$\psi_\Sigma(\sigma_i) = t, \quad \psi_\Sigma(\zeta_j) = q, \quad \psi_\Sigma(\bar{\sigma}_m) = 1$$

and

$$\psi_\Sigma(\mu_r) = m_r, \quad \psi_\Sigma(\lambda_r) = \ell_r, \quad \psi_\Sigma(\bar{\mu}_r) = \bar{m}_r, \quad \psi_\Sigma(\bar{\lambda}_r) = \bar{\ell}_r$$

where $1 \leq i \leq k-1, 1 \leq j \leq n, 1 \leq m \leq n-1$ and $1 \leq r \leq g$. If $k=1$, then we define H_Σ to be the quotient of the above group by $t=1$. Then H_D is isomorphic to G_D defined earlier for all $k \geq 1$, and is a subgroup of H_Σ for any Σ and $k \geq 1$. Even though H_Σ (or H_D) depends on whether $k=1$ or $k \geq 2$, our notation does not indicate it for the sake of simplicity.

Let $\phi_\Sigma : \mathbf{B}_{n,k}(\Sigma) \rightarrow G_\Sigma$ be the restriction of ψ_Σ to $\mathbf{B}_{n,k}(\Sigma)$ onto G_Σ that denotes the normal subgroup $\psi_\Sigma(\mathbf{B}_{n,k}(\Sigma))$ of H_Σ generated by $\{q, t, m_1, \dots, m_g, \ell_1, \dots, \ell_g\}$. Then we can find the covering $p : \tilde{B}_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$ corresponding to $\text{Ker } \phi_\Sigma$. Since the braid group $\mathbf{B}_{0,n}(\Sigma)$ embeds into the mapping class group of punctured surface Σ_n , a braid $\beta \in \mathbf{B}_{0,n}(\Sigma)$ determines a homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$. Recall that the induced homomorphism $\bar{\beta}_*$ on $\mathbf{B}_{n,k}(\Sigma)$ is in fact the same as the conjugation by $\iota(\beta)$ where $\iota : \mathbf{B}_{0,n}(\Sigma) \rightarrow \mathbf{B}_{n,k}(\Sigma)$ is the splitting map in Lemma 2.3.

Lemma 3.1. *Using the notations above, the homeomorphism $\bar{\beta} : B_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$ lifts to a homeomorphism $\tilde{\beta} : \tilde{B}_{n,k}(\Sigma) \rightarrow \tilde{B}_{n,k}(\Sigma)$ for any $\beta \in \mathbf{B}_{0,n}(\Sigma)$ and the restriction ϕ_Σ of ψ_Σ satisfies $\beta_\# \phi_\Sigma = \phi_\Sigma \bar{\beta}_*$*

Proof. By the lifting criteria, $\bar{\beta}$ lifts to $\tilde{\beta}$ if and only if $\bar{\beta}_*(\text{Ker } \phi_\Sigma) \subset \text{Ker } \phi_\Sigma$ if and only if there is an induced automorphism $\beta_\#$ on G_Σ given by $\beta_\# \phi_\Sigma = \phi_\Sigma \bar{\beta}_*$. Thus it suffices to show that $\phi_\Sigma \bar{\beta}_*(W) = 1$ for any $W \in \text{Ker } \phi_\Sigma$ and $\beta \in \mathbf{B}_{0,n}(\Sigma)$. Let W be a word in generators $\{\mu_i, \lambda_i, \sigma_i, \zeta_i\}$ of $\mathbf{B}_{n,k}(\Sigma)$. Since the presentation for H_Σ shows that any two elements are commutative up to multiplications by central elements q and t , we have

$$\phi_\Sigma(W) = W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = q^c t^d \prod m_i^{a_i} \ell_i^{b_i}$$

where $W(\{x_i \leftarrow y_i\})$ denote the word obtained from W by replacing the generators x_i 's by y_i 's.

Suppose $\phi_\Sigma(W) = 1$. Then $a_i = b_i = 0$ for all $1 \leq i \leq g$. Thus for generators $\sigma_r, \mu_r, \lambda_r$ for $\mathbf{B}_{0,n}(\Sigma)$, we have

$$\begin{aligned} \phi_\Sigma((\bar{\sigma}_r)_*(W)) &= \phi_\Sigma(W(\zeta_r \leftarrow \zeta_r \zeta_{r+1} \zeta_r^{-1}, \zeta_{r+1} \leftarrow \zeta_r)) \\ &= W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1, \\ \phi_\Sigma((\bar{\mu}_r)_*(W)) &= \phi_\Sigma(W(\lambda_r \leftarrow \lambda_r \mu_r \zeta_1^{-1} \mu_r^{-1})) \\ &= W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \lambda_r \leftarrow q^{-1} \ell_r, \sigma_i \leftarrow t, \zeta_i \leftarrow q) \\ &= q^{-b_r} W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1, \end{aligned}$$

$$\begin{aligned} \text{and } \phi_\Sigma((\bar{\lambda}_r)_*(W)) &= \phi_\Sigma(W(\mu_r \leftarrow \lambda_r \zeta_1 \lambda_r^{-1} \mu_r \zeta_1^{-1} \lambda_r^{-1})) \\ &= W(\mu_i \leftarrow m_i, \mu_r \leftarrow q m_r, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) \\ &= q^{a_r} W(\mu_i \leftarrow m_i, \lambda_i \leftarrow \ell_i, \sigma_i \leftarrow t, \zeta_i \leftarrow q) = 1. \end{aligned}$$

Therefore $\phi_\Sigma(\bar{\beta}_*(W)) = 1$ and so $\beta_\#(\phi_\Sigma(\alpha)) = \phi_\Sigma(\bar{\beta}_*(\alpha))$ holds for all $\alpha \in \mathbf{B}_{n,k}(\Sigma)$. \square

By the above lemma, we now have a \mathbb{Z} -module automorphism $\tilde{\beta}_*$ on $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$. Notice that $\tilde{\beta}_*$ is not necessarily a $\mathbb{Z}[G_\Sigma]$ -module homomorphism since the condition $(*)$ in §2.1 may not hold, that is, the automorphism $\beta_\#$ of G_Σ may not be the identity.

On the other hand, $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{B}_{n,k}(\Sigma)$ by the right multiplication and so there is an induced action of β on H_Σ given by $\beta \cdot h = h\psi_\Sigma(\beta)$ for $h \in H_\Sigma$. It is possible to alter the induced action by multiplying with a certain function χ from $\mathbf{B}_{0,n}(\Sigma)$ to the centralizer of G_Σ in H_Σ . We will closely discuss this possibility in Theorem 4.4.

Using the \mathbb{Z} -module automorphism $\tilde{\beta}_*$ and the action on $\mathbf{B}_{n,k}(\Sigma)$ by $\mathbf{B}_{0,n}(\Sigma)$, we construct a $\mathbb{Z}[H_\Sigma]$ -module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H_\Sigma] \otimes_{\mathbb{Z}[G_\Sigma]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ by

$$(\beta \otimes \tilde{\beta}_*)(h \otimes c) = (\beta \cdot h) \otimes \tilde{\beta}_*(c)$$

for $h \in H_\Sigma$ and $c \in H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$.

Theorem 3.2. *Let Σ be a compact, connected, oriented 2-dimensional manifold with non-empty boundary. Let H_Σ be the group (depending on k) and $\psi_\Sigma : \mathbf{B}_{n,k}(\Sigma) \rightarrow H_\Sigma$ be the epimorphism given in the discussion above and let $\phi_\Sigma : \mathbf{B}_{n,k}(\Sigma) \rightarrow G_\Sigma = \phi_\Sigma(\mathbf{B}_{n,k}(\Sigma))$ be the restriction of ψ_Σ . Then there is a homomorphism*

$$\Phi_k : \mathbf{B}_{0,n}(\Sigma) \rightarrow \text{Aut}_{\mathbb{Z}[H_\Sigma]} \left(\mathbb{Z}[H_\Sigma] \otimes_{\mathbb{Z}[G_\Sigma]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma)) \right)$$

defined by

$$\Phi_k(\beta) = \beta \otimes \tilde{\beta}_*$$

where the action of β on H_Σ is given by $\beta \cdot h = h\psi_\Sigma(\beta)$ for $h \in H_\Sigma$.

Moreover, this family Φ_k of representation is an extension of homology linear representation of the classical braid group $\mathbf{B}_{0,n}(D)$ in the sense of Definition 2.2.

Proof. Clearly Φ_k is a group homomorphism. To see the well-definedness and the $\mathbb{Z}[H_\Sigma]$ -linearity of $\Phi_k(\beta)$, we assert that

$$\beta \cdot (hh') = (\beta \cdot h)\beta_\#(h').$$

for all $h \in H_\Sigma, h' \in G_\Sigma$. Then

$$\begin{aligned} (\beta \otimes \tilde{\beta}_*)(h \otimes h'c) &= (\beta \cdot h) \otimes \tilde{\beta}_*(h'c) = (\beta \cdot h) \otimes \beta_\#(h')\tilde{\beta}_*(c) \\ &= (\beta \cdot h)\beta_\#(h') \otimes \tilde{\beta}_*(c) = (\beta \otimes \tilde{\beta}_*)(hh' \otimes c) = hh'(\beta \otimes \tilde{\beta}_*)(1 \otimes c) \end{aligned}$$

for $c \in H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$. Here the last equality is clear by the definition of the action by β .

To show the assertion, choose $\alpha \in \mathbf{B}_{n,k}(\Sigma)$ such that $\phi_\Sigma(\alpha) = h'$. By Lemma 3.1, we have

$$\beta_\#(\phi_\Sigma(\alpha)) = \phi_\Sigma(\bar{\beta}_*(\alpha)) = \psi_\Sigma(\bar{\beta}_*(\alpha)) = \psi_\Sigma(\beta^{-1}\alpha\beta) = \psi_\Sigma(\beta)^{-1}\phi_\Sigma(\alpha)\psi_\Sigma(\beta)$$

Thus

$$\beta \cdot (hh') = hh'\psi_\Sigma(\beta) = (h\psi_\Sigma(\beta))(\psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta)) = (\beta \cdot h)\beta_\#(h').$$

To show that Φ_k is an extension of homology linear representations of the classical braid groups, we regard an n punctured disk D_n as a subspace of Σ_n . Then the configuration space $B_{n,k}(D)$ is a subspace of the configuration space $B_{n,k}(\Sigma)$. For

the covering $p : \tilde{B}_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$ corresponding to ϕ_Σ , a connected component of $p^{-1}(B_{n,k})$ is denoted by $\tilde{B}_{n,k}(D)$. Since G_D embeds into G_Σ , $p|_{\tilde{B}_{n,k}(D)} : \tilde{B}_{n,k}(D) \rightarrow B_{n,k}(D)$ is the covering over $B_{n,k}(D)$ corresponding to $\psi|_{\mathbf{B}_{n,k}(D)} : \mathbf{B}_{n,k}(D) \rightarrow G_D$. In fact, $H_k^{BM}(\tilde{B}_{n,k}(D))$ is a submodule of $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ as $\mathbb{Z}[G_D]$ -modules. One can see this more explicitly in the proof of Lemma 3.3. Each braid $\beta \in \mathbf{B}_{0,n}(D)$ gives a $\mathbb{Z}[H_D]$ -module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H_D] \otimes_{\mathbb{Z}[G_D]} H_k^{BM}(\tilde{B}_{n,k}(D))$. Since $G_D = H_D$, this automorphism is the same as $\tilde{\beta}_*$ on $H_k^{BM}(\tilde{B}_{n,k}(D))$, which is the homology linear representation of the classical braid group. \square

In §4, we will show that if one wants to obtain the result similar to the above theorem, the extension $\mathbf{B}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ is determined uniquely up to redundant coefficient extension, and the quotient H_Σ is uniquely determined for $k \geq 3$ and is the simplest for $k \geq 1$ in the sense that any proper quotient of H_Σ does not contain G_D properly.

3.2. Computation of proposed representations. We now compute an explicit matrix forms of the representations described in Theorem 3.2, which turn out to be extensions of the Burau representation and Lawrence-Krammer-Bigelow representation of the classical braid groups. The following lemma and its proof show not only that $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ is a free $\mathbb{Z}[G_\Sigma]$ -module but also how to choose a basis. The lemma is an extension of the corresponding lemma on a disk by Bigelow [9] and we borrow the main idea of his proof.

Lemma 3.3. *The homology group $H_\ell^{BM}(\tilde{B}_{n,k}(\Sigma))$ is the direct sum of*

$$\binom{2g+n+k-2}{k}$$

copies of $\mathbb{Z}[G_\Sigma]$ for $\ell = k$ and trivial otherwise.

Proof. Let d be a metric on Σ that can be either hyperbolic or Euclidean. Suppose punctures z_1^0, \dots, z_n^0 lie on a geodesic. And let γ_j be the geodesic segment joining z_j^0 and z_{j+1}^0 for $1 \leq j \leq n-1$. For $1 \leq i \leq g$, let α_i, β_i be geodesic loops based at z_1^0 that represent the meridian and the longitude of the i -th handle so that α_i 's, β_i 's, and γ_j 's are mutually disjoint. Let Γ be the union of all of these arcs so that Γ_n is a disjoint union of open $2g+n-1$ geodesic segments. Consider

$$B_\Gamma = B_{n,k}(\Gamma) = \{\{z_1, \dots, z_k\} \subset \Gamma_n\}.$$

Then it is not hard to see B_Γ is homeomorphic to a disjoint union of $\binom{2g+n+k-2}{k}$ open k -balls that can be parameterized by $(2g+n-1)$ -tuples (r_1, \dots, r_{2g+n-1}) of nonnegative integers that add up to k so that the i -th segment of Γ_n contains r_i points from $\{z_1, \dots, z_k\}$.

Let $p : \tilde{B}_{n,k}(\Sigma) \rightarrow B_{n,k}(\Sigma)$ be the covering corresponding to the epimorphism $\phi_\Sigma : B_{n,k}(\Sigma) \rightarrow G_\Sigma$. We will be done if we show that

$$H_\ell^{BM}(p^{-1}(B_\Gamma)) \rightarrow H_\ell^{BM}(\tilde{B}_{n,k}(\Sigma))$$

induced by the inclusion is an isomorphism since $H_\ell^{BM}(B_\Gamma)$ is isomorphic to the direct sum of $\binom{2g+n+k-2}{k}$ copies of $H_\ell(D^k, S^{k-1})$.

Define the family of compact subsets A_ϵ of Σ_n defined by

$$A_\epsilon = \{\{z_1, \dots, z_k\} \in B_{n,k}(\Sigma) \mid d(z_i, z_j) \geq \epsilon \text{ for } i \neq j, d(z_i, z_j^0) \geq \epsilon \text{ for all } i, j\}.$$

Since any compact subset of $B_{n,k}(\Sigma)$ is contained in A_ϵ for sufficiently small $\epsilon > 0$, it suffices to show that

$$H_\ell(p^{-1}(B_\Gamma), p^{-1}(B_\Gamma - A_\epsilon)) \rightarrow H_\ell(\tilde{B}_{n,k}(\Sigma), p^{-1}(B_{n,k}(\Sigma) - A_\epsilon))$$

is an isomorphism.

Let $\Sigma_\epsilon \subset \Sigma$ be the closed ϵ neighborhood of Γ and let $B_\epsilon = B_{n,k}(\Sigma_\epsilon)$. Then the obvious homotopy collapsing from $B_{n,k}(\Sigma)$ to B_ϵ gives the isomorphism

$$H_\ell(p^{-1}(B_\epsilon), p^{-1}(B_\epsilon - A_\epsilon)) \rightarrow H_\ell(\tilde{B}_{n,k}(\Sigma), p^{-1}(B_{n,k}(\Sigma) - A_\epsilon)).$$

Let B be the set of $\{x_1, \dots, x_k\} \in B_\epsilon$ such that for each x_i there exists the unique nearest point in Γ_n . Then B is open and contains $A_\epsilon \cap B_\epsilon$. By excision, the inclusion induces an isomorphism

$$H_\ell(p^{-1}(B), p^{-1}(B - A_\epsilon)) \rightarrow H_\ell(p^{-1}(B_\epsilon), p^{-1}(B_\epsilon - A_\epsilon)).$$

Finally, the obvious deformation retract from B to B_Γ gives an isomorphism

$$H_\ell(p^{-1}(B), p^{-1}(B - A_\epsilon)) \rightarrow H_\ell(p^{-1}(B_\Gamma), p^{-1}(B_\Gamma - A_\epsilon)).$$

that completes the proof. \square

We remark that $B_\epsilon = B_{n,k}(\Sigma_\epsilon)$ and $B_\Gamma = B_{n,k}(\Gamma)$ do not have the same homotopy type even though Γ is a deformation retract of Σ_ϵ . This is because Γ is a 1-dimensional complex and movements of points in Γ avoiding collision are more restricted.

Let $I(n, k, g)$ be the set of $(2g + n - 1)$ -tuples of nonnegative integers that add up to k . The proof of Lemma 3.3 shows that a typical basis for $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ as a free $\mathbb{Z}[G_\Sigma]$ -module can be indexed by the set $I(n, k, g)$. The proof also shows that a basis for $H_k^{BM}(\tilde{B}_{n,k}(D))$ can be chosen as a subset of a basis for $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$. Thus the homology linear representations for $\mathbf{B}_{0,n}(D)$ appears in matrix forms of proposed representations for $\mathbf{B}_{0,n}(\Sigma)$ as minors.

By the work of Krammer [22, 23] and Bigelow [6, 8, 9], it has been known that there is a natural and useful way of describing a basis geometrically. Recall the loops α_i, β_i and the arcs γ_j from the proof of Lemma 3.3. For $(r_1, \dots, r_{2g+n-1}) \in I(n, k, g)$, choose r_i disjoint duplicates of α_i or β_{i-g} or γ_{i-2g} if $1 \leq i \leq g$ or $g+1 \leq i \leq 2g$ or $2g+1 \leq i \leq 2g+n-1$, respectively. For each i , join these r_i disjoint duplicates to $\partial\Sigma$ by mutually disjoint arcs (that determine a basing). This geometric object is called a *fork*. In fact, a fork uniquely determines a k -cycle in $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ by lifting the Cartesian product of k curves together with basing arcs in the fork. Notice that the basing is required to have a unique lift. For example, the fork corresponding to $(0, 2, 1, 0, 0, 3) \in I(3, 6, 2)$ looks like the set of curves on the left of Figure 1.

As Bigelow showed for the case of the disk in [9], the Poincaré duality, the universal coefficient theorem, and Lemma 3.3 imply that the ordinary relative homology $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ is the dual space of the Borel-Moore homology $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ in the sense that there is a nonsingular sesquilinear pairing

$$\langle \cdot, \cdot \rangle : H_k^{BM}(\tilde{B}_{n,k}(\Sigma)) \times H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma)) \rightarrow \mathbb{Z}[S]$$

where S is a skew field containing $\mathbb{Z}[G_\Sigma]$. In fact, the group G_Σ is bi-ordered and so it can embed into a skew field such as the Malcev-Neumann power series

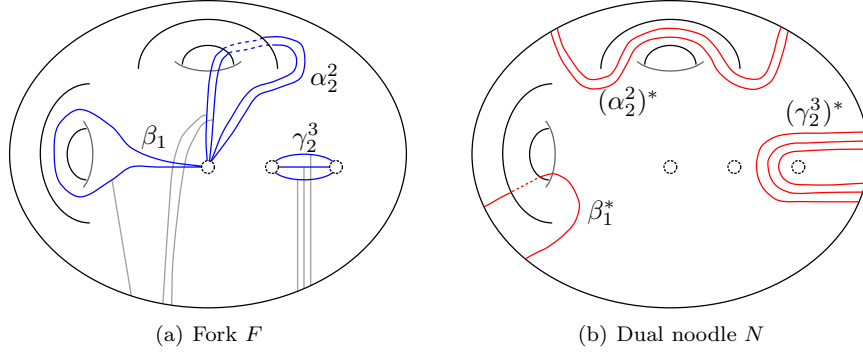


FIGURE 1. An example of a fork and its dual noodle

ring [26, 27]. Explicitly, the above pairing is defined as follows: for cycles $F \in H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$, $N \in H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ in each homology groups

$$\langle F, N \rangle = \sum_{y \in G_\Sigma} y(F, yN)$$

where (\cdot, \cdot) counts the intersection number.

Let $\alpha_1^*, \dots, \alpha_g^*, \beta_1^*, \dots, \beta_g^*, \gamma_1^*, \dots, \gamma_g^*$ be pairwise disjoint arcs which start and end at $\partial\Sigma$ and α_i^* (or β_i^* , or γ_j^*) intersects only α_i (or β_i , or γ_j) once transversely. A basis of $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ by duplicating α_i^* 's or β_i^* 's or γ_j^* 's depending on a given $(2g + n - 1)$ -tuples in $I(n, k, g)$. This geometric object is called a *noodle*. In fact, a noodle uniquely determines a relative k -cycle in $H_k(\tilde{B}_{n,k}(\Sigma), \partial\tilde{B}_{n,k}(\Sigma))$ by lifting the Cartesian product of k arcs in the noodle. For example, the noodle corresponding to $(0, 2, 1, 0, 0, 3) \in I(3, 6, 2)$ looks like the set of arcs on the right of Figure 1.

For a k -cycle F determined by a fork, and a relative k -cycle N determined by a noodle, the sesquilinear pairing $\langle F, N \rangle$ computes algebraic intersections between them with $\mathbb{Z}[G_\Sigma]$ coefficients. This pairing can easily be computed by recording intersections between the fork and the noodle on Σ . The basis determined by forks and the basis determined by noodles are dual with respect to the pairing.

In the case of a disc, Bigelow [9] showed that this pairing is invariant under the action by $\mathbf{B}_{0,n}(D)$. But in the case of a surface Σ of genus ≥ 1 , it can not be invariant under the action by $\mathbf{B}_{0,n}(\Sigma)$. In fact, the pairing cannot be preserved by any braid group action given by a representation Ψ into $\text{Aut}_{\mathbb{Z}[G_\Sigma]}(H_k^{BM}(\tilde{B}_{n,k}))$. Suppose it is preserved, that is, for any $\beta \in \mathbf{B}_{0,n}(\Sigma)$, a k -cycle F determined by a fork, and a relative k -cycle N determined by a noodle,

$$\langle F, N \rangle = \langle \Psi(\beta)(F), \Psi(\beta)(N) \rangle,$$

then for any $y \in G_\Sigma$,

$$\begin{aligned} y\langle F, N \rangle &= \langle yF, N \rangle \\ &= \langle \Psi(\beta)(yF), \Psi(\beta)(N) \rangle \\ &= \beta_\#(y) \langle \Psi(\beta)(F), \Psi(\beta)(N) \rangle \\ &= \beta_\#(y) \langle F, N \rangle. \end{aligned}$$

The property $y = \beta_{\sharp}(y)$ for all $y \in G_{\Sigma}$ forces to set $q = 1$ in G_{Σ} and so it was abandoned.

Nonetheless, we can extend this pairing to

$\langle \cdot, \cdot \rangle_{H_{\Sigma}} : \mathbb{Z}[H_{\Sigma}] \otimes_{\mathbb{Z}[G_{\Sigma}]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma)) \times \mathbb{Z}[H_{\Sigma}] \otimes_{\mathbb{Z}[G_{\Sigma}]} H_k(\tilde{B}_{n,k}(\Sigma), \partial \tilde{B}_{n,k}(\Sigma)) \rightarrow S'$
defined by

$$\left\langle \sum_i g_i F_i, \sum_j h_j N_j \right\rangle_{H_{\Sigma}} = \sum_{i,j} g_i \langle F_i, N_j \rangle h_j^{-1}$$

where $g_i, h_j \in \mathbb{Z}[H_{\Sigma}]$ and S' is the skew field containing $\mathbb{Z}[H_{\Sigma}]$. Note that this extended pairing cannot be invariant under the braid group action given by Φ_k either. But it can be used to compute proposed representations Φ_k explicitly. The following theorem summarizes the above discussion.

Theorem 3.4. *Let F_i 's and N_i 's be k -cycles and relative k -cycles in dual bases determined by forks and noodles. Then for each $\beta \in \mathbf{B}_{0,n}(\Sigma)$, $\Phi_k(\beta)$ is represented by a matrix with respect to the basis $\{F_i \mid 1 \leq i \leq \binom{2g+n+k-2}{k}\}$ whose (i, j) -th entry is given by*

$$\psi_{\Sigma}(\beta) \left\langle \tilde{\beta}(F_i), N_j \right\rangle_{H_{\Sigma}}$$

which is an element of $\mathbb{Z}[H_{\Sigma}]$ rather than of S' .

As an example, we will show the matrix form of the representation Φ_1 of the 3-braid group $\mathbf{B}_{0,3}(\Sigma)$ which is an extension of the Burau representation where $\Sigma = \Sigma(2, 1)$. Since $k = 1$, the basis of $H_1^{BM}(\tilde{B}_{3,1}(\Sigma))$ determined by forks can be expressed by $\{\gamma_1, \gamma_2, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ and similarly the dual basis of $H_1(\tilde{B}_{3,1}(\Sigma), \partial \tilde{B}_{3,1}(\Sigma))$ determined by noodles is written by $\{\gamma_1^*, \gamma_2^*, \alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*\}$.

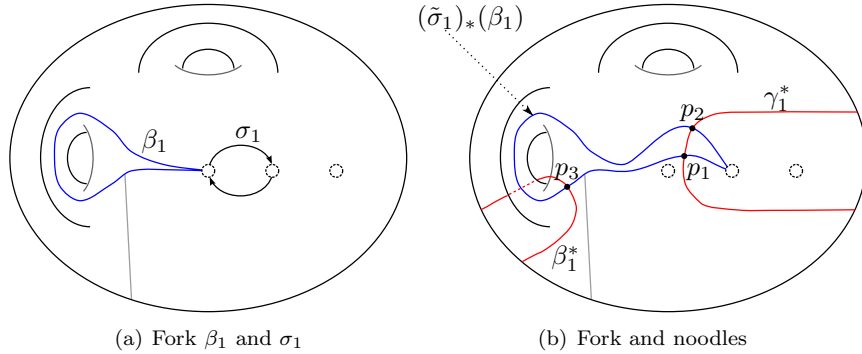


FIGURE 2. An example of the pairing of a fork and a noodle

Figure 2 shows the action of σ_1 on the fork β_1 . Also it shows intersection points p_1, p_2 with γ_1^* and p_3 with β_1^* . In the covering space, the intersection point p_1 lies on the sheet transformed by q since the fork wraps a puncture, and p_2 lies on the sheet transformed by $\ell_1 q$ since the fork contains the longitude of the first handle and wraps a puncture. Besides we have the negative sign for p_2 since the orientation is switched. Finally p_3 lies on the sheet containing the base point of the covering space. Therefore we have $\Phi_1(\sigma_1)(\beta_1) = \beta_1 + q(1 - \ell_1)\gamma_1$. By the similar computation, we can obtain every entry of $\Phi_1(\sigma_1)$.

Note that the correspondence between the basis $\{v_{j,k}\}$ in [6] and our basis is given as follows:

$$\begin{aligned} v_{1,2} &= -tq^{-4}w_{1,1} \\ v_{1,3} &= -tq^{-4}(w_{1,1} + q(1 - t^{-1})w_{1,2} + q^2w_{2,2}) \\ v_{2,3} &= -tq^{-2}w_{2,2} \end{aligned}$$

Then the action of Φ_2 on the basis $\{v_{j,k}\}$ together with substitution $t \mapsto -t$ is exactly equal to that of Lawrence-Krammer-Bigelow representation in [6].

4. JUSTIFICATION OF PROPOSED REPRESENTATION

Beside the family of representations proposed in the previous section, we will investigate the possibility that there may be other representations of the surface braid groups that extends the homology linear representations of the classical braid groups. One may try to consider alternatives in the three ways—a group extension of $\mathbf{B}_{n,k}(\Sigma)$ other than $\mathbf{B}_{n;k}(\Sigma)$, a quotient group of $\mathbf{B}_{n;k}(\Sigma)$ other than H_Σ , and an action on H_Σ by $\mathbf{B}_{0,n}(\Sigma)$ other than the right multiplication via the quotient map.

4.1. Group extension of $\mathbf{B}_{n,k}(\Sigma)$. In order to make an adjustment of coefficients in the most flexible manner, we may try to find a group extension $\mathbf{E}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ such that $\mathbf{B}_{0,n}(\Sigma)$ acts on $\mathbf{E}_{n,k}(\Sigma)$ and it is as large as possible. If we regard $\mathbf{B}_{0,n}(\Sigma)$ and $\mathbf{B}_{n,k}(\Sigma)$ as subgroups of some large braid group $\mathbf{B}_{0,n+k+\ell}(\Sigma)$, then $\mathbf{B}_{0,n}(\Sigma)$ acts naturally on $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ as well as on $\mathbf{B}_{n,k}(\Sigma)$ by conjugation. Thus we assume that $\mathbf{B}_{n,k}(\Sigma) \subset \mathbf{E}_{n,k}(\Sigma) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$ for some $\ell \geq 0$.

Lemma 4.1. *Let Σ be a surface with non-empty boundary and Σ' be a collar neighborhood of $\partial\Sigma$. Let $N(\mathbf{B}_{n,k}(\Sigma))$ denote the normalizer of $\mathbf{B}_{n,k}(\Sigma)$ in $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ for some $\ell \geq 0$. Then*

$$N(\mathbf{B}_{n,k}(\Sigma)) \cong \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma').$$

Proof. We first identify $\mathbf{B}_{n,k}(\Sigma)$ and $\mathbf{B}_{0,n}(\Sigma)$ to the corresponding subgroups of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ via the embeddings which add trivial ℓ and $k+\ell$ strands, respectively. Then we will show $N(\mathbf{B}_{n,k}(\Sigma)) = \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma')$ as subgroups of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$. It is clear that $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma') \subset N(\mathbf{B}_{n,k}(\Sigma))$ since $\mathbf{B}_{n,k}(\Sigma)$ is a normal subgroup of $\mathbf{B}_{n;k}(\Sigma)$ from the short exact sequence of Lemma 2.3 and elements of $\mathbf{B}_{0,\ell}(\Sigma')$ commutes with those of $\mathbf{B}_{n,k}(\Sigma)$. Conversely let $\beta \in N(\mathbf{B}_{n,k}(\Sigma)) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$. Any element $\alpha \in \mathbf{B}_{n,k}(\Sigma)$ and its conjugate $\beta^{-1}\alpha\beta \in \mathbf{B}_{n,k}(\Sigma)$ induce permutations that preserve the sets $\{1, \dots, n\}$, $\{n+1, \dots, n+k\}$ and $\{n+k+1, \dots, n+k+\ell\}$. It is easy to see that the induced permutation of β itself must fix the above three sets since α can be arbitrary in $\mathbf{B}_{n,k}(\Sigma)$. Thus $\beta \in \mathbf{B}_{n+k;\ell}(\Sigma)$ and the split exact sequence

$$1 \longrightarrow \mathbf{B}_{n+k;\ell}(\Sigma) \longrightarrow \mathbf{B}_{n+k;\ell}(\Sigma) \xrightarrow{(\pi_{n+k})_*} \mathbf{B}_{0,n+k}(\Sigma) \longrightarrow 1$$

gives a unique decomposition $\beta = \beta_1\beta_2$ for $\beta_1 \in \mathbf{B}_{0,n+k}(\Sigma)$ and $\beta_2 \in \mathbf{B}_{n+k;\ell}(\Sigma)$. In fact, $\beta_1 = (\pi_{n+k})_*(\beta) \in \mathbf{B}_{n;k}(\Sigma)$ since the epimorphism $(\pi_{n+k})_*$ forgets the last ℓ strands or replaces them by the trivial ℓ -strand braid.

For any $\alpha \in \mathbf{B}_{n,k}(\Sigma) \subset \mathbf{B}_{0,n+k}(\Sigma) \subset \mathbf{B}_{0,n+k+\ell}(\Sigma)$, we have $(\pi_{n+k})_*(\beta_2^{-1}\alpha\beta_2) = \beta_2^{-1}\alpha\beta_2$ since $\beta_2^{-1}\alpha\beta_2 \in \mathbf{B}_{0,n+k}$. On the other hand, we have $(\pi_{n+k})_*(\beta_2^{-1}\alpha\beta_2) =$

α since $(\pi_{n+k})_*$ replaces the last ℓ strands by the trivial braid. Thus we have $\beta_2^{-1}\alpha\beta_2 = \alpha$. From the presentation of $\mathbf{B}_{0,n+k+\ell}(\Sigma)$ in §1.1, it is easy to see that $\beta_2 \in \mathbf{B}_{n+k,\ell}(\Sigma)$ must be a local braid in order for β_2 to commute with every element of $\mathbf{B}_{n,k}(\Sigma)$. Thus we have $\beta_2 \in \mathbf{B}_{0,\ell}(\Sigma')$ where Σ' is an annulus that is a collar neighborhood of $\partial\Sigma$ in Σ . Consequently, we have shown $N(\mathbf{B}_{n,k}(\Sigma)) \subset \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,\ell}(\Sigma')$ \square

By the above lemma, the extension $\mathbf{E}_{n,k}(\Sigma)$ of $\mathbf{B}_{n,k}(\Sigma)$ can be taken as a subgroup of $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,l}(\Sigma')$. We remark that $\mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}_{0,l}(\Sigma')$ is also a subgroup of the intertwining braid group $\mathbf{B}_{n;k+l}(\Sigma)$.

Then we follow the construction given in the discussion before Theorem 3.2 with $\mathbf{E}_{n,k}(\Sigma)$ replacing $\mathbf{B}_{n,k}(\Sigma)$.

Let $\psi : \mathbf{E}_{n,k}(\Sigma) \rightarrow H$ be an epimorphism onto a group H . If we choose an action of $\mathbf{B}_{0,n}(\Sigma)$ on the extension $\mathbf{E}_{n,k}(\Sigma)$ then the action is carried over H via ψ and it is convenient to use the convention that $(\beta_1\beta_2) \cdot h = \beta_2 \cdot (\beta_1 \cdot h)$ for $\beta_1, \beta_2 \in \mathbf{B}_{0,n}(\Sigma)$ and $h \in H$. In order to obtain a $\mathbb{Z}[H]$ -module automorphism $\beta \otimes \tilde{\beta}_*$ on $\mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ that is an extension of a homology linear representation of the classical braid group, this induced action of $\mathbf{B}_{0,n}(\Sigma)$ on H needs to satisfy the following two conditions:

- (i) Lifting criteria : $\beta_{\#}$ exists and $\beta_{\#}(\phi(\alpha)) = \phi(\tilde{\beta}_*(\alpha))$ for all $\alpha \in \mathbf{B}_{n,k}(\Sigma)$ where $\phi = \psi|_{\mathbf{B}_{n,k}(\Sigma)}$;
- (ii) Linearity and compatibility : $hh'(\beta \cdot 1) = \beta \cdot (hh') = (\beta \cdot h)\beta_{\#}(h')$ for all $h \in H, h' \in G = \phi(\mathbf{B}_{n,k}(\Sigma))$.

As in the proof of Theorem 3.2, we then have

$$(\beta \otimes \tilde{\beta}_*)(h \otimes h'c) = (\beta \otimes \tilde{\beta}_*)(hh' \otimes c) = hh'(\beta \otimes \tilde{\beta}_*)(1 \otimes c)$$

for all $h \in H, h' \in G$ and $c \in H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$.

Theorem 4.2. *Suppose that there are an epimorphism $\psi : \mathbf{E}_{n,k}(\Sigma) \rightarrow H$ and an action of $\mathbf{B}_{0,n}(\Sigma)$ on H satisfying the above two conditions. Let Ψ_k be the representation obtained from ψ and the action. Then*

$$\Psi_k = 1_{\mathbb{Z}[H]} \otimes_{\mathbb{Z}[H']} \Psi'_k$$

for a representation Ψ'_k obtained from an epimorphism $\psi' : \mathbf{B}_{n;k}(\Sigma) \rightarrow H' \subset H$ and an action $\mathbf{B}_{0,n}(\Sigma)$ on H' where $1_{\mathbb{Z}[H]}$ is the identity map on $\mathbb{Z}[H]$.

Proof. Let $H' = \{\beta \cdot 1 \in H \mid \beta \in \mathbf{B}_{0,n}(\Sigma)\} \phi(\mathbf{B}_{n,k}(\Sigma))$ and $\psi' : \mathbf{B}_{n;k}(\Sigma) \rightarrow H'$ be a surjection defined by $\psi'(\beta) = \beta \cdot 1$ for $\beta \in \mathbf{B}_{0,n}(\Sigma)$ and $\psi' = \phi$ on $\mathbf{B}_{n,k}(\Sigma)$. Then since

$$\psi'(\beta_1\beta_2) = (\beta_1\beta_2) \cdot 1 = \beta_2 \cdot (\beta_1 \cdot 1) = (\beta_1 \cdot 1)(\beta_2 \cdot 1) = \psi'(\beta_1)\psi'(\beta_2)$$

for all $\beta_1, \beta_2 \in \mathbf{B}_{0,n}(\Sigma)$, ψ' becomes a homomorphism and ψ' preserves the semidirect product structure. Also we have

$$\phi' = \psi'|_{\mathbf{B}_{n,k}(\Sigma)} = \psi|_{\mathbf{B}_{n,k}(\Sigma)} = \phi$$

and so ϕ and ϕ' induce the same homology group $H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ and two \mathbb{Z} -module automorphisms obtained from β coincide.

Now consider two representations Ψ_k and Ψ'_k corresponding to ψ and ψ' , respectively. Then $\Psi_k(\beta)$ gives a $\mathbb{Z}[H]$ -homomorphism on $\mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ and $\Psi'_k(\beta)$ gives a $\mathbb{Z}[H']$ -homomorphism on $\mathbb{Z}[H'] \otimes_{\mathbb{Z}[G]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$. Since $\mathbb{Z}[H] =$

$\mathbb{Z}[H] \otimes_{\mathbb{Z}[H']} \mathbb{Z}[H']$, $\Psi_k(\beta)$ is a $\mathbb{Z}[H]$ -homomorphism on $\mathbb{Z}[H] \otimes_{\mathbb{Z}[H']} \mathbb{Z}[H'] \otimes_{\mathbb{Z}[G]} H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$ defined by

$$\Psi_k(\beta)(hh' \otimes c) = hh'(\beta \cdot 1) \otimes \tilde{\beta}_*(c) = h \otimes h'(\beta \cdot 1) \otimes \tilde{\beta}_*(c)$$

for all $h \in \mathbb{Z}[H]$, $h' \in \mathbb{Z}[H']$ and $c \in H_k^{BM}(\tilde{B}_{n,k}(\Sigma))$. This is equal to $1_{\mathbb{Z}[H]} \otimes_{\mathbb{Z}[H']} \Psi'_k(\beta)$. \square

The above theorem implies that we may assume that $\mathbf{B}_{n;k}(\Sigma) \subset \mathbf{E}_{n,k}(\Sigma)$ without loss of generality. Then by Lemma 4.1, $\mathbf{E}_{n,k}(\Sigma) = \mathbf{B}_{n;k}(\Sigma) \times \mathbf{B}$ for some subgroup \mathbf{B} of $\mathbf{B}_{0,\ell}(\Sigma')$ and the above theorem says that any family of representations obtained by using $\mathbf{E}_{n,k}(\Sigma)$ is merely a trivial extension of the family of representations proposed in §3.

4.2. Quotient of $\mathbf{B}_{n;k}(\Sigma)$. According to the scheme described in Theorem 3.2, it is important to find a good epimorphism $\psi : \mathbf{B}_{n;k}(\Sigma) \rightarrow H$ onto some group H .

Since Σ is not sphere, the inclusion $B_{n,k}(D) \hookrightarrow B_{n,k}(\Sigma)$ induces a monomorphism $\mathbf{B}_{n,k}(D) \hookrightarrow \mathbf{B}_{n,k}(\Sigma)$ (See [11]). Similarly, the inclusion $B_{n,k}(D) \hookrightarrow B_{n,k}(\Sigma)$ induces a monomorphism $\mathbf{B}_{n;k}(D) \hookrightarrow \mathbf{B}_{n;k}(\Sigma)$ which will be regarded as an inclusion.

We first determine an epimorphism $\psi_D : \mathbf{B}_{n;k}(D) \rightarrow H_D$ to extend the map $\phi_D : \mathbf{B}_{n,k}(D) \rightarrow G_D$ for the classical braid groups. Since we want to obtain the homology linear representations for the classical braid groups, we should use the $H_D = G_D$ and all of extra generators $\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}$ for $\mathbf{B}_{n;k}(D)$ should be sent to the identity by ψ_D as we have seen earlier in §3.1. Then $\psi_D|_{\mathbf{B}_{n,k}(D)} = \phi_D$. For some extension H of G_D , let $\psi : \mathbf{B}_{n;k}(\Sigma) \rightarrow H$ be an epimorphism. In order to obtain an extension of homology linear representations of the classical braid groups via ψ , the following condition is required:

$$(**) \quad \psi|_{\mathbf{B}_{n;k}(D)} = \psi_D$$

$$\begin{array}{ccccc} \mathbf{B}_{n,k}(D) & \hookrightarrow & \mathbf{B}_{n;k}(D) & \hookrightarrow & \mathbf{B}_{n;k}(\Sigma) \\ \downarrow \phi_D & & \downarrow \psi_D & & \downarrow \psi \\ G_D & \xlongequal{\quad} & H_D & \hookrightarrow & H \end{array}$$

FIGURE 3. Quotients of braid groups on a disk D and surface Σ

This condition is nothing but a reinterpretation of Definition 2.2 and is necessary to make the diagram in Figure 3 commutative so that $\psi|_{\mathbf{B}_{n,k}(D)} = \phi_D$ and we can apply the construction of Theorem 3.2. We first show that $(**)$ uniquely determines the quotient group H for $k \geq 3$.

Theorem 4.3. *Suppose $k \geq 3$. Let $\psi : \mathbf{B}_{n;k}(\Sigma) \rightarrow H$ be an arbitrary epimorphism onto a group H satisfying $(**)$. Then H is isomorphic to H_Σ defined in §3.1.*

Proof. Let $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \rightarrow H_\Sigma$ be the epimorphism defined in §3.1. We want to show $\psi : \mathbf{B}_{n;k}(\Sigma) \rightarrow H$ factor through H_Σ , that is, there is an epimorphism $h : H_\Sigma \rightarrow H$ such that $h\psi_\Sigma = \psi$.

Recall the presentation for $\mathbf{B}_{n;k}(\Sigma)$ in Lemma 2.5. The condition $(**)$ implies $\psi(\sigma_i) = q, \psi(\zeta_j) = t$, and $\psi(\bar{\sigma}_m) = 1$ for all $1 \leq i \leq k-1, 1 \leq j \leq n$, and $1 \leq m \leq n-1$. Since $k \geq 3$, the relation CR1 among generators in X_2 is not vacuous and so the relation CR1 through CR3 for X_2 and the condition $(**)$ imply

$$[\psi(\mu_r), q] = [\psi(\lambda_r), q] = [\psi(\mu_r), t] = [\psi(\lambda_r), t] = [q, t] = 1$$

for all $1 \leq r \leq g$. Also the relation (iii) in Lemma 2.5 implies

$$[\psi(\bar{\mu}_r), q] = [\psi(\bar{\lambda}_r), q] = [\psi(\bar{\mu}_r), t] = [\psi(\bar{\lambda}_r), t] = 1$$

for all $1 \leq r \leq g$. Thus q and t lie in the center of H . Using this, all other relations in H_Σ can be shown to hold in H . Therefore ψ induces an epimorphism $h : H_\Sigma \rightarrow H$.

To prove that h is an isomorphism, it suffices to show that for any word W in generators of H_Σ , $h(W) = 1$ implies $W = 1$.

Using the relations of H_Σ , we may assume that W is a word of the following form:

$$W = q^c t^d \prod_{i=1}^g m_i^{a_i} \ell_i^{b_i} \bar{m}_i^{\bar{a}_i} \bar{\ell}_i^{\bar{b}_i}$$

And this is equivalent to

$$m_r^{a_r} = W W_r^{-1} \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1}$$

where $W_r = \left(q^c t^d \prod_{i \neq r} m_i^{a_i} \ell_i^{b_i} \bar{m}_i^{\bar{a}_i} \bar{\ell}_i^{\bar{b}_i} \right)$. The relation $[m_r, \bar{\ell}_r] = q$ implies $m_r^{a_r} \bar{\ell}_r = q^{a_r} \bar{\ell}_r m_r^{a_r}$. Then for $1 \leq r \leq g$,

$$W W_r^{-1} \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1} \bar{\ell}_r = m_r^{a_r} \bar{\ell}_r = q^{a_r} \bar{\ell}_r m_r^{a_r} = q^{a_r} \bar{\ell}_r W W_r^{-1} \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1}.$$

Now apply h and use $h(W) = 1$ to have:

$$\begin{aligned} h(W_r^{-1}) h \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1} h(\bar{\ell}_r) &= h(q^{a_r} \bar{\ell}_r) h(W_r^{-1}) h \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1} \\ &= h(W_r^{-1}) h(q^{a_r}) h \left(\ell_r^{b_r} \bar{m}_r^{\bar{a}_r} \bar{\ell}_r^{\bar{b}_r} \right)^{-1} h(\bar{\ell}_r) \end{aligned}$$

Thus $h(q^{a_r}) = 1$. Since q is of infinite order by $(**)$, $a_r = 0$. Similarly, we can show that $b_r = \bar{a}_r = \bar{b}_r = 0$ in turn. Then the condition $(**)$ implies $c = d = 0$. Consequently, $W = 1$. \square

For $k \leq 2$, the condition $(**)$ cannot uniquely determine a quotient group of $\mathbf{B}_{n;k}(\Sigma)$. To take advantage of representations in analyzing the surface braid group $\mathbf{B}_{0,n}(\Sigma)$, one may prefer a simpler coefficient ring as long as representations carries enough information. For the classical case, there are also several groups satisfying the condition $(*)$ unless we assume abelian. For the surface braid groups, we cannot obtain any interesting representation if an abelian coefficient ring is used as we discussed in §2. Thus it is natural to require a kind of minimality for the choice of a quotient H of $\mathbf{B}_{n;k}(\Sigma)$ in the sense that any further quotient of H violates the condition $(**)$. Then H_Σ given in §3.1 is minimal for all k and n . The proof of this minimality is similar to that of Theorem 4.3. Consequently, Theorem 4.3 forces us to use the epimorphism $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \rightarrow H_\Sigma$.

We now discuss possible actions of $\mathbf{B}_{0,n}(\Sigma)$ on H_Σ induced from ψ_Σ .

Theorem 4.4. *Let $\psi_\Sigma : \mathbf{B}_{n;k}(\Sigma) \rightarrow H_\Sigma$ be the epimorphism defined in §3.1. Let $\beta \cdot h$ denote any action on $h \in H_\Sigma$ by $\beta \in \mathbf{B}_{0,n}(\Sigma)$ that is induced from ψ_Σ and satisfies the two conditions given above Theorem 4.2. Then*

$$\beta \cdot h = h\chi(\beta)\psi_\Sigma(\beta)$$

for some function $\chi : \mathbf{B}_{0,n}(\Sigma) \rightarrow C_{H_\Sigma}(G_\Sigma)$ with the property that $(\chi, \psi_\Sigma) : \mathbf{B}_{0,n}(\Sigma) \rightarrow C_{H_\Sigma}(G_\Sigma) \rtimes H_\Sigma$ is a homomorphism where $C_{H_\Sigma}(G_\Sigma)$ denotes the centralizer of G_Σ in H_Σ .

Proof. By the hypotheses of the action, we have

$$h'(\beta \cdot 1) = \beta \cdot (1h') = (\beta \cdot 1)\beta_\#(h')$$

and

$$\beta_\#(h') = \psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta)$$

for all $h' \in G_\Sigma$. By combining two equations, we have

$$\psi_\Sigma(\beta)^{-1}h'\psi_\Sigma(\beta) = (\beta \cdot 1)^{-1}h'(\beta \cdot 1).$$

and so $(\beta \cdot 1)\psi_\Sigma(\beta)^{-1} \in C_{H_\Sigma}(G_\Sigma)$. Hence $(\beta \cdot 1) = \chi(\beta)\psi_\Sigma(\beta)$ for a function $\chi : \mathbf{B}_{0,n}(\Sigma) \rightarrow C_{H_\Sigma}(G_\Sigma)$. Since $\chi(\beta_1\beta_2)\psi_\Sigma(\beta_1\beta_2) = (\beta_1\beta_2) \cdot 1 = \beta_2 \cdot (\beta_1 \cdot 1) = (\chi(\beta_1)\psi_\Sigma(\beta_1))\chi(\beta_2)\psi_\Sigma(\beta_2)$, we have

$$\chi(\beta_1\beta_2) = \chi(\beta_1)\psi_\Sigma(\beta_1)\chi(\beta_2)\psi_\Sigma(\beta_1)^{-1}.$$

This implies that

$$\begin{aligned} (\chi(\beta_1\beta_2), \psi_\Sigma(\beta_1\beta_2)) &= (\chi(\beta_1)\psi_\Sigma(\beta_1)\chi(\beta_2)\psi_\Sigma(\beta_1)^{-1}, \psi_\Sigma(\beta_1\beta_2)) \\ &= (\chi(\beta_1), \psi_\Sigma(\beta_1))(\chi(\beta_2), \psi_\Sigma(\beta_2)). \end{aligned}$$

Therefore $(\chi, \psi_\Sigma) : \mathbf{B}_{0,n}(\Sigma) \rightarrow C_{H_\Sigma}(G_\Sigma) \rtimes H_\Sigma$ is a homomorphism. \square

The function χ in the above theorem behaves like a character of $\mathbf{B}_{0,n}(\Sigma)$. In fact, if $k \geq 2$, then it can be shown that $C_{H_\Sigma}(G_\Sigma) = Z(H_\Sigma) = \langle q \rangle \oplus \langle t \rangle$. Hence χ can be any homomorphism from $\mathbf{B}_{0,n}(\Sigma)$ to $Z(H_\Sigma)$. In this case, the representations Ψ_k obtained from ψ is given by $\Psi_k = \chi \otimes \Phi_k$ for some character χ where Φ_k is the proposed representation in Theorem 3.2.

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